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Internal hydraulic jumps and mixing in two-layer flows

By DAVID M. HOLLAND[1], RODOLFO R. ROSALES[2], DAN STEFANICA[2] AND ESTEBAN G. TABAK[1]

[1]Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012-1185, USA

[2]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Internal hydraulic jumps in two-layer flows are studied, with particular emphasis on their role in entrainment and mixing. For highly entraining internal jumps, a new closure is proposed for the jump conditions. The closure is based on two main assumptions: (i) most of the energy dissipated at the jump goes into turbulence, and (ii) the amount of turbulent energy that a stably stratified flow may contain without immediately mixing further is bounded by a measure of the stratification. As a consequence of this closure, surprising bounds emerge, for example on the amount of entrainment that may take place at the location of the jump. These bounds are probably almost achieved by highly entraining internal jumps, such as those likely to develop in dense oceanic overflows. The values obtained here are in good agreement with the existing observations of the spatial development of oceanic downslope currents, which play a crucial role in the formation of abyssal and intermediate waters in the global ocean.

1. Introduction

Internal shocks are ubiquitous occurrences in the atmosphere and the ocean. Conventionally, they are denoted bores when they propagate into a state largely at rest, and hydraulic jumps when they consist of a standing discontinuity within a nearly steady flow. Examples in the atmosphere are bores associated with dense inversion layers propagating over topographic obstacles (Dobrinski *et al.* 2001). Examples in the ocean are the internal hydraulic jumps likely to develop in dense overflows downstream of a sill, such as: the overflow of Mediterranean Waters over the strait of Gibraltar (O'Neil Baringer & Price 1997*a, b*); the overflow of Arctic Waters over the Denmark Strait; and the overflow of Antarctic Bottom Water through the Vema Channel separating the Argentinian and Brazilian basins (Hogg 1993). Actual hydraulic jumps have been inferred from observations and numerical simulations only for a few of these dense overflows; more detailed observations in the near future should reveal whether they are present in all of them (See also Nash & Moum (2001) for a clear observation of a hydraulic jump over a smaller scale obstacle within the continental shelf of Oregon.) Gravity currents (Benjamin 1968; Simpson 1997) are also limiting cases of bores, in which the state ahead of the bore is largely homogeneous. Simple experiments with two-layer fluids show that internal jumps develop as naturally in stratified flows as they do in free-surface flows of shallow, homogeneous single layers (Long 1954; Wood & Simpson 1984). This can be seen analytically too, though the mathematics and physics become subtler as one switches from discrete layers to a continuously stratified profile.

Very little is known, however, about the nature and structure of internal shocks. Even in the simplest possible scenario of two-layer flows, the problem remains wide open. Conservation of mass and momentum still yield two jump conditions. However, if fluid from one layer is entrained into the other at the jump, a new unknown appears (the amount of entrainment) for which no physical conservation principle is readily available. Existing literature (Armi 1986; Baines 1995; Klemp, Rotunno & Skamarock 1997; Lawrence 1990, 1993; Wood & Simpson 1984; Yih & Guha 1955) concentrates mostly on the case of jumps that do not involve any mixing. Pawlak & Armi (2000) present interesting experimental work on fluid entrainment in downslope flows downstream of a hydraulically controlled sill. However, they arrange their experimental setting so that the flow is supercritical throughout the slope, thus excluding shocks.

A few words are appropriate here about the nature of mixing in oceanic flows. In the bulk of the global ocean, vertical (diapycnal) turbulent mixing is usually modelled as being diffusive in nature, with a diffusivity about two orders of magnitude above the molecular one (Ledwell *et al.* 2000). The energy behind this turbulent diffusivity often comes from sources which are external to the particular layers being mixed; tides are an ubiquitous example of such an external source. A description of this externally driven diffusivity is provided in classical work by Ellison & Turner (1959) and Turner (1986); see also Balmforth, Llewellyn Smith & Young (1998) for a more recent approach. However, particularly violent mixing phenomena, such as strong downslope flows, provide their own source of energy. In this context, mixing is generally thought to occur through shear instability (Howard 1961; Miles 1961). This idea has been incorporated into large-scale ocean models through parameterizations such as Hallberg's (2000), which switch from externally supplied to self-driven mixing, based on the local Richardson number. Still, this framework implies mixing only over relatively extended spatial domains. However, if internal hydraulic jumps develop within downslope flows, they must involve a considerable amount of strongly localized, self-supplied mixing.

This paper focuses on highly entraining standing internal hydraulic jumps, such as those likely to occur in permanent dense overflows. These are different from internal bores propagating into a state at rest (Klemp *et al.* 1997; Lane-Serff & Woodward 2001), in that the sign of the vorticity at the interface between the lower layer of dense fluid and the ambient is such that it favours entrainment of ambient fluid into the lower layer. For strong jumps, it is conjectured that this entrained fluid will be rapidly mixed throughout the layer by the strong turbulence generated at the jump, thus rendering the lower layer nearly uniform, both in density and in velocity.

For concreteness, here we will look at the simplest setting for internal hydraulic jumps: a 'one-and-a-half-layer' flow. This is a two-layer flow constrained by upper and lower rigid lids, in which the height of the upper layer is very large, the density difference between the two layers is small, and all mixing occurs through the entrainment of upper fluid into the lower layer. Even in its simplicity, this scenario is not unrealistic for many flows of geophysical significance, such as dense overflows over sills. The density changes brought about by the entrainment process make developing a closure for the jump conditions a non-trivial matter. The closure proposed here is based on the energetic considerations described in §§ 3 and 4. In fact, we develop two closures. The first one is a *partial closure*. It does not consist exclusively of jump conditions, but also of a jump inequality, arising from a bound on the amount of

			b	u	h	F_1	F_2
Partial Closure	Lower bound for b	$d=2$	**0.588**	0.358	4.738	3.953	0.848
		$d=3$	**0.508**	0.298	6.589	5.054	0.824
	Lower bound for u	$d=2$	1	**0.229**	4.365	3.422	0.375
		$d=3$	1	**0.182**	5.486	4.218	0.328
	Upper bound for h	$d=2$	0.654	0.274	**5.572**	4.082	0.585
		$d=3$	0.567	0.226	**7.772**	5.279	0.567
Full Closure	Lower bound for b	$d=2$	**0.641**	0.382	4.079	3.457	0.816
		$d=3$	**0.547**	0.313	5.824	4.537	0.797
	Lower bound for u	$d=2$	1	**0.267**	3.732	2.971	0.412
		$d=3$	1	**0.208**	4.791	3.724	0.355
	Upper bound for h	$d=2$	0.701	0.303	**4.705**	3.576	0.596
		$d=3$	0.603	0.245	**6.767**	4.734	0.574
	Upper bound for F_1	$d=2$	0.673	0.318	4.658	**3.595**	0.647
		$d=3$	0.581	0.255	6.721	**4.751**	0.615
	Lower bound for F_1	$d=2$	1	1	1	**1.892**	1.892
		$d=3$	1	1	1	**2.149**	2.149
	Upper bound for F_2	$d=2$	1	1	1	1.892	**1.892**
		$d=3$	1	1	1	2.149	**2.149**
	Lower bound for F_2	$d=2$	1	0.267	3.732	2.971	**0.412**
		$d=3$	1	0.208	4.791	3.724	**0.355**

TABLE 1. Bounds for b, u, h, the ratios (downstream over upstream) of the values for the buoyancy, velocity and height of the lower layer, respectively; and for F_1 and F_2, the Froude numbers upstream and downstream; d is the dimensionality of the turbulence, either 2 or 3. The bounds found are in bold font; the values of the remaining variables when the bounds are achieved are in regular font.

turbulent energy that a stratified flow may possess. The second one is a *full closure*, obtained by turning the bound on the turbulent energy into an equality. This is justified for highly entraining internal hydraulic jumps that are maximally turbulent both upstream and downstream of the jump.

Our closure hypothesis can be summarized in a few words: that most of the energy dissipated in strong, entraining internal hydraulic jumps goes into turbulence. Then, a bound on the amount of dissipation follows easily, since a stratified fluid can only develop a certain amount of turbulence, without being mixed further. Both the partial and the full closures yield bounds on the various quantities involved in the jump. These bounds are in fact quite surprising: we bound the amount of allowable entrainment, the Froude numbers of the flow before and after the jump, and the height and velocity ratios between the two sides of the jump – bounds with analogues in gas dynamics, but not in hydraulics. We believe that these bounds are realized by turbulent hydraulic jumps, such as those likely to appear in oceanic dense overflows.

The bounds that we found are summarized in table 1. There, b, u and h denote the ratios (downstream over upstream) of the values for the buoyancy, velocity and height of the lower layer, respectively; F_1 and F_2 are the Froude numbers upstream and downstream; and d is the dimensionality of the turbulence, assumed to be either two-dimensional ($d = 2$) or three-dimensional ($d = 3$). Real flows should lie somewhere in between these two extremes. The numbers in bold font are the bounds found, while those in regular font give the values of the remaining variables when the bounds are achieved. The bounds are split into two sections, corresponding to the partial closure and to the full closure, respectively.

The lower bounds on b are particularly relevant, since they imply that the volume flow of dense fluid should roughly double across a strong internal hydraulic jump. This result is consistent with observations of the Mediterranean outflow, which roughly doubles within the Gulf of Cadiz (O'Neil Baringer & Price 1997*a*), and of the Denmark Strait overflow, which also doubles within a few hundred kilometres downstream of the sill (Saunders 2001). We conjecture that such doubling of the flow should also take place for Antarctic Bottom Waters on their way from the Antarctic continental shelf to the ocean floor. Currently, there are not enough observations available to validate this conjecture, but large-scale expeditions are planned for the near future that will provide the needed data.

The reason why we obtain bounds instead of precise numbers, even for the full closure, is that we are not assuming any knowledge about the states upstream or downstream of the jump. In fact, once the Froude number upstream F_1 is specified, we can make precise predictions; these are reported in § 6.3, and constitute an ideal benchmark to test our theory against experiments.

Finally, it is physically clear that no hydraulic jumps should be possible without dissipation. It is interesting to note that this result follows easily from our mathematical formulation; see §§ 3 and 5 for more details.

The rest of the paper is structured as follows. In § 2, we introduce the one-and-a-half-layer model, describe the closure problem, and discuss existing closures in the literature. In §§ 3 and 4, we introduce our partial closure hypothesis: an inequality based on energetic considerations involving turbulent flows. In § 5, we explore the consequences of this hypothesis, and find the resulting bounds on all dynamical quantities, including the amount of entrainment. In § 6, we explore the consequences of a stronger hypothesis: that the flow is maximally turbulent both upstream and downstream of the jump. This hypothesis leads to sharp predictions, that can be validated against observations and experiments. Finally, in § 7 we discuss the validity and relevance of the results obtained.

2. The simplest setting

In this section, we describe what we believe to be the simplest scenario that captures the closure problem for entraining hydraulic jumps: a two-dimensional, 'one-and-a-half' layer model, under the Boussinesq approximation.

Since we envision applications of our work to oceanic flows, some words are in order regarding the effects of rotation on internal breaking waves. There is a relatively widespread belief in the geophysical community that rotation inhibits shock formation (see, however, Houghton 1969; Pratt, Hefrich & Chassignet 2000). One of the reasons why shocks are rarely thought of in a geophysical context is a misguided analogy with the Rossby adjustment problem, in which an initial discontinuity resembling a shock is allowed to evolve in a rotating environment, leading to the establishment of a smooth steady state in geostrophic balance. This appears to indicate that shocks are not robust objects in the presence of fast rotation. Yet this is a wrong interpretation since, even without rotation, the Rossby adjustment problem – which becomes the dam breaking problem of hydraulics – is dominated by a strong rarefaction wave, with a much weaker leading shock. A more general reason why shocks are largely neglected is that they do not appear in the quasi-geostrophic (QG) approximation, which is frequently used to model flows with fast rotation. However, one must remember that the QG approximation filters all gravity waves, by assuming the relevant scales to be long and the velocities weak. Hence, shocks are excluded from consideration *ad initio*.

Mathematically, one may argue that non-differentiated terms, such as the Coriolis

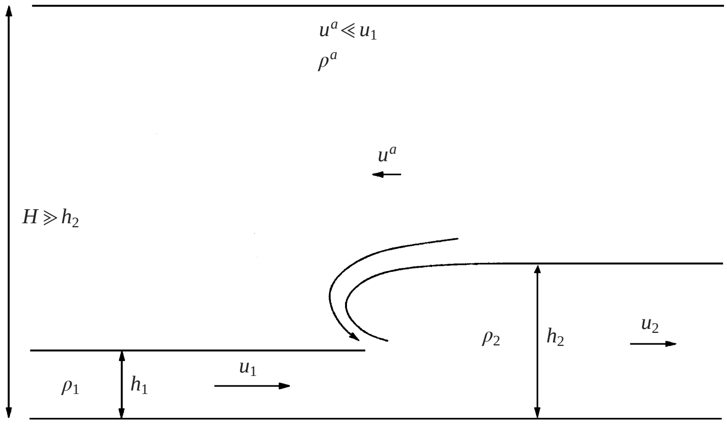

FIGURE 1. Two-layer flow with entraining internal hydraulic jump.

acceleration, cannot affect the local behaviour at shocks. On more physical grounds, one should compare an estimate for the width of a hydraulic jump with a typical value for the internal radius of deformation of the ocean. Since the former ranges in the hundreds of metres, and the latter in the tens of kilometres, one may safely conclude that the effects of rotation on the jump are, if not negligible, certainly far from dominant. This is not to say, of course, that rotation does not play a fundamental role in establishing the flow in the context of which the jump may occur. In particular, downslope flows in a rotating environment tend to attach themselves to lateral boundaries. As far as the jump conditions are concerned, however, the role of rotation is clearly minor.

In order to introduce the model, consider the configuration depicted in figure 1, consisting of a standing internal hydraulic jump within a two-layer flow, in a channel with rigid top and bottom lids, separated by a distance H. Since the top layer will be assumed to be much deeper than the bottom one, and thought of as an ambient, we will use the superindex a to identify the corresponding variables, such as the velocity and density, while no superindices will be used for the variables representing quantities associated with the bottom, active layer. Thus h, ρ and u represent the height, density and velocity of the bottom layer, while $H - h$, ρ^a and u^a represent the same quantities for the top, ambient layer. We will denote by P the pressure at the top rigid lid. Since we only consider hydraulic jumps in which all entrainment takes place from the ambient fluid into the bottom layer, the ambient density ρ^a is a constant, while ρ is not. Subindices 1 and 2 are used to denote the values of the corresponding variables to the left and right of the jump respectively (that is, upstream and downstream of the jump). As we will see below, in the limit of a very deep ambient, the system reduces to a simple one, involving only quantities associated with the bottom layer.

Three constraints are immediate for the variables involved in the jump: global conservation of mass, horizontal momentum and volume; the last since the flow is assumed to be incompressible. Conservation of volume yields

$$[hu + (H - h)u^a] = 0. \tag{2.1}$$

Here and throughout this paper, the brackets stand for the jump of the enclosed expression across the shock, i.e. the difference between its values downstream and

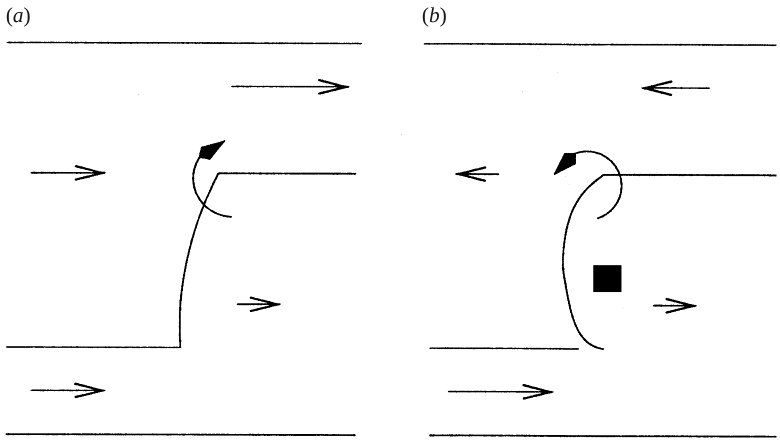

FIGURE 2. Distinction between internal bores (a) and hydraulic jumps (b). The velocities, represented by straight arrows, are drawn in a frame of reference in which the shock waves do not move. The curved arrows show the sign of the vorticity at the interface between the two layers, favourable to entrainment of ambient fluid only for the hydraulic jump.

upstream. Conservation of mass yields

$$[\rho hu + \rho^a(H-h)u^a] = 0. \tag{2.2}$$

Equations (2.1) and (2.2) can be combined into one for 'conservation of buoyancy',

$$[bhu] = 0, \tag{2.3}$$

where b is the reduced gravity or buoyancy

$$b = \frac{\rho - \rho^a}{\rho^a} g, \tag{2.4}$$

and g is the acceleration due to gravity.

In order to write down the equation for global conservation of momentum, we need to compute the momentum flux and the vertical integral of the pressure on both sides of the jump. The former is given by

$$\rho hu^2 + \rho^a(H-h)(u^a)^2,$$

while the latter, using the hydrostatic approximation, takes the form

$$PH + g\rho^a\{\tfrac{1}{2}(H-h)^2 + (H-h)h\} + \tfrac{1}{2}g\rho h^2 = \tfrac{1}{2}g\rho^a H^2 + PH + \tfrac{1}{2}\rho^a bh^2.$$

Hence global conservation of momentum yields

$$[\rho hu^2 + \rho^a(H-h)(u^a)^2 + PH + \tfrac{1}{2}\rho^a bh^2] = 0. \tag{2.5}$$

Up to here, we have only made the following assumptions: vertically uniform flow within each layer, hydrostatic balance away from the jump, and that all entrainment takes place from the upper to the lower layer, with the entrained fluid rapidly mixed throughout the latter. At this point, however, we need to make stronger assumptions, in order to reach closure.

First, we need to clarify the distinction between a bore and a hydraulic jump (Baines 1995; Klemp *et al.* 1997). Schematic representations of both are depicted in figure 2, in frames of reference moving with the shocks. For the bore, which is in reality moving to the left into an area of quiescent flow, this choice of frame of reference implies that the velocities upstream of the bore are equal in sign and magnitude. Then, downstream, volume conservation implies that the velocities are still equal in sign, but that the velocity of the lower, expanding layer is smaller than that of the upper layer. By contrast, the hydraulic jump has velocities of opposite sign in the two layers, both upstream and downstream of the jump. This corresponds, in the geophysical situation of dense overflows, to light ambient water flowing near the surface against the deep overflow, in order to replenish its source. In Klemp *et al.* (1997), it is argued that the resulting signs of the vortex sheets at the interface between the two layers is such that it favours entrainment of ambient fluid into the lower layer for hydraulic jumps, but not for bores. Thus, for hydraulic jumps, most of the energy dissipation takes place in the lower, expanding layer – a situation analogous to that of external, single-layer jumps. On the other hand, the expanding layer may even experience an energy increase across internal bores (Klemp *et al.* 1997).

In this paper, we consider internal hydraulic jumps. This is consistent with our hypothesis of exclusively downward entrainment. It also implies a closure for the pressure P at the top rigid lid. Since there is no or little dissipation in the upper layer, P satisfies, at least approximately, Bernoulli's principle:

$$[\tfrac{1}{2}\rho^a(u^a)^2 + P] = 0. \tag{2.6}$$

Some words are in order about this closure, since variations of it have occupied most of the discussion in the theoretical literature on internal shocks to date. The approaches range from an *ad hoc* closure for the form drag between the two layers in Yih & Guha (1955), to the mutually exclusive assumptions that all energy dissipation takes place in either the lower (Wood & Simpson 1984) or the upper layers (Klemp *et al.* 1997). The closure proposed in Klemp *et al.* (1997) is the most adequate for bores. However, for the internal hydraulic jumps considered in this paper, one should enforce a condition similar to the one proposed in Wood & Simpson (1984), though without the inconsistencies brought about there by the lack of a distinction between bores and hydraulic jumps, and the neglect of the effects of mixing on the density of the lower layer.

Next, we will make the approximation that the ambient layer is much deeper than the bottom one. This approximation simplifies the mathematics of the problem significantly, and it is also quite realistic for most instances of geophysical dense overflows. Equation (2.1) can be integrated to

$$hu + (H - h)u^a = Q, \tag{2.7}$$

where Q is the volume flow through the shock. For hydraulic jumps, Q is arguably small, or at least bounded, even as H gets very large. In the context of dense overflows, $-Q$ represents the amount of water at the source of the overflow lost due to evaporation, as in the Mediterranean and Red Seas, or to freezing, as in the Antarctic continental shelf. Hence, for hydraulic jumps, the assumption that $H \gg h$ implies that $|u^a| \ll |u|$. This allows us to neglect the second term in (2.5), and, using (2.6), also the third term.

In order to simplify equation (2.5) even further, we notice that, in most oceanic applications, ρ and ρ^a are very close to each other. Thus, we can make the Boussinesq approximation, and replace ρ in the first term of (2.5) by ρ^a. This leads to the following, much reduced, form of the momentum equation:

$$[hu^2 + \tfrac{1}{2}bh^2] = 0. \tag{2.8}$$

In cases where the buoyancy b does not change across the hydraulic jump, i.e. if there is no significant entrainment, equations (2.3) and (2.8) provide the necessary pair of jump conditions for the two dynamical variables u and h. These jump conditions are in fact the same as the standard ones for external hydraulic jumps, with reduced gravity b. Essentially, this is the approach taken in Klemp *et al.* (1997), Wood & Simpson (1984) and Yih & Guha (1955), with qualifications given by their various closures for the pressure and their consideration of the case with finite ambient depth. This approach makes sense for bores, where the sign of interfacial vorticity does not favour entrainment. However, for internal hydraulic jumps of the kind studied here, entrainment at the jump could be considerable, affecting significantly the buoyancy of the fluid. Hence we need one more equation – or, rather, one more physical principle – for closure. A proposal on how to fill this gap, and an exploration of its consequences, constitutes the rest of this paper.

3. Energetic considerations

Where should we look for the missing physical principle needed to close the system of jump conditions for internal hydraulic jumps? It is typical in mechanics that, once mass and momentum have been considered, one searches for missing clues in the principles of conservation of angular momentum and energy. Both principles play important roles at internal hydraulic jumps: the former, in setting the vorticity of the jump's main roll, as well as the torque of the non-hydrostatic component of the pressure; the latter, in determining the jump's irreversibility, through energy transfer from the well-ordered mean flow, to highly unorganized turbulent and eventually thermal motion. The issues arising from considerations of angular momentum will be described in Holland & Tabak (2002); see also Valiani (1997). Here, we concentrate on energetic considerations, which we believe are crucial in determining the main properties of internal hydraulic jumps.

Energy considerations are not newcomers to jump conditions in fluid systems. It is instructive to notice how different a role they play in regular hydraulics versus gas dynamics. In both systems, mass and momentum conservation for standing shocks take a form nearly identical to that in (2.3) and (2.8). In hydraulics,

$$[hu] = 0, \quad [\rho h u^2 + p] = 0,$$

where p is the hydrostatic pressure, integrated vertically over the water height h, and ρ is the constant density of water. In gas dynamics,

$$[\rho u] = 0, \quad [\rho u^2 + p] = 0,$$

where u is the fluid velocity, ρ is the (variable) density, and p is the pressure. In both cases, we can add an energy equation, i.e.

$$[\tfrac{1}{2}\rho u^3 + pu + \rho u e] = 0$$

for gas dynamics, and

$$[\tfrac{1}{2}\rho h u^3 + \tfrac{1}{2}g\rho h^2 u + pu + \rho h u e] = 0 \tag{3.1}$$

for hydraulic jumps. The slight difference arises from the existence of a potential energy in hydraulics, which has no gas-dynamical analogue. The new variable e represents the internal energy of the gas and, in hydraulics, all forms of energy not accounted for by the mean flow. These are usually conceptualized as mostly thermal, but, in reality, have a strong turbulent component, in addition to the surface and gravitational energy residing in the air bubbles entrained into the flow. For undular hydraulic jumps, there is also a wave radiation component to the energy. This wave-energy is not modelled well by equation (3.1), since it is not associated with water parcels and therefore does not travel at the mean speed of the fluid.

There is a far more significant difference between hydraulics and gas dynamics. In the former, the integrated pressure p is a function of the height h. Hence the equations for mass and momentum constitute a closed system, and the energy equation can be used as a diagnostic for the amount of energy dissipated. In gas dynamics, on the other hand, the pressure p is a function of the density ρ *and* the internal energy e, through the equation of state. Thus the energy equation is strictly required to close the system. Our proposal for internal hydraulic jumps lies somewhere in between these two extreme cases. We relate p and e through an inequality instead of an equation of state, and use it as a diagnostic tool. For highly entraining flows, we turn this inequality into an equality, thus obtaining a full closure.

In order to focus the discussion, we write the system of equations (2.3) and (2.8), together with an energy equation involving the yet unspecified 'internal energy' density e:

$$[bhu] = 0, \tag{3.2}$$

$$[hu^2 + \tfrac{1}{2}bh^2] = 0, \tag{3.3}$$

$$[\tfrac{1}{2}hu^3 + bh^2u + hue] = 0. \tag{3.4}$$

The derivation of the energy equation (3.4) follows the same pattern as that of the momentum equation, under the assumption that the internal energy e, consisting of all forms of energy not accounted for by the mean flow or by the potential energy, is transported by the fluid. This excludes from our discussion the undular jumps, where radiation of wave energy plays an important role. It also excludes the – necessarily weak – jumps where much of the energy goes into organized, as opposed to turbulent, shear. Our contention is that, for strong enough jumps, turbulent mixing homogenizes the flow, destroying most organized shear, as well as suppressing most radiating waves. We restrict our discussion to jumps satisfying this condition.

The system (3.2)–(3.4) could be closed by specifying an 'equation of state' relating b, h and e, if this made physical sense. Before going in this direction, we explore how the energy equation can help to close the system, by considering first the case with no energy dissipation, i.e. $[hue] = 0$. We expect no hydraulic jump to be possible under such conditions, since hydraulic jumps dissipate energy. Proving this statement has independent interest; it will also help us obtain the algebra involved in extracting meaningful information from the highly nonlinear system (3.2)–(3.4).

We index the variables upstream of the jump with 1, those downstream with 2, and introduce the non-dimensional ratios

$$u = \frac{u_2}{u_1}, \qquad h = \frac{h_2}{h_1}, \qquad b = \frac{b_2}{b_1}, \tag{3.5}$$

and the Froude numbers

$$F_j = \frac{|u_j|}{\sqrt{b_j h_j}}. \tag{3.6}$$

We also note that hydraulic jumps must satisfy the 'entropy' conditions

$$u \leqslant 1, \qquad h \geqslant 1, \qquad b \leqslant 1. \tag{3.7}$$

The last of these follows from the fact that the fluids can only mix, not 'unmix', at the jump. The other two are standard in regular hydraulics: that the flow decelerates and expands across the jump follows from the the condition that the flow needs to switch from a supercritical to a subcritical regime.

Notice that we use the following notational convention, that should not introduce confusion: Within the context of discussing conditions at hydraulic jumps, u, h and b always denote the non-dimensional ratios above. When discussing general equations applying to the flow, u, h and b denote, respectively, the flow velocity in the bottom layer, the height of the bottom layer, and the buoyancy, as defined in (2.4).

The jump conditions (3.2)–(3.4), together with $[hue] = 0$, can be rewritten as

$$bhu = 1,$$

$$1 + \frac{1}{2}\frac{1}{F_1^2} = h\,u^2 + \frac{1}{2}\frac{h}{u}\frac{1}{F_1^2},$$

$$1 + \frac{2}{F_1^2} = hu^3 + h\frac{2}{F_1^2}.$$

The last two equations can be combined to eliminate F_1:

$$\tfrac{3}{2} - 2h + \frac{h}{2u} - 2hu^2 + \tfrac{3}{2}h^2u^2 + \tfrac{1}{2}hu^3 = 0. \tag{3.8}$$

The statement that no hydraulic jump is possible without energy dissipation is equivalent to:

for $h \geqslant 1$ and $u \leqslant 1$, equation (3.8) has no solution other than $h = u = 1$.

To prove this, we note that the same result should hold on inverting h, u and b (after all, with no dissipation, there is no particular difference between up- and downstream). Introducing the symmetrized variables

$$x = b + \frac{1}{b} = hu + \frac{1}{hu}, \qquad y = u + \frac{1}{u},$$

we can recast (3.8) as

$$3(x-2) + (y-2)^2 = 0.$$

From their definitions, $x, y \geqslant 2$, and therefore $x = y = 2$, corresponding to $b = h = u = 1$. This concludes the proof.

In § 5, we will show that not only $[hue] \neq 0$ for internal jumps, but in fact $[hue] > 0$. In other words, internal energy must be generated at a hydraulic jump, as one would expect in any dissipative process.

4. A partial closure for the energy

In order to develop a closure, we need to make some assumptions on the nature of the internal energy e. What is this energy composed of? By taking the two fluids to be miscible, as is the case in most geophysical applications, we exclude any energy from going into surface tension. We have already excluded wave and organized shear energy, by assuming the flow immediately downstream of the jump to be highly turbulent and hence well-mixed, both vertically (suppressing shear) and horizontally (averaging out waves). The two main remaining forms of energy are thermal and the kinetic energy of the turbulence itself. We now argue that, in the neighbourhood of strong hydraulic jumps, the latter dominates over the former (see Rouse, Sato & Nagaratnam 1959 for a classical account). Given the small viscosity of water, in order for the mechanical energy of the incoming flow to be dissipated into heat, it needs to cascade through a long inertial range of scales. In this inertial range, the flow is essentially inviscid, and can best be described as turbulent. Eventually, most of the turbulent energy cascades down to the dissipation range, and becomes thermal. However, this takes far more time than the fluid spends in a neighbourhood of the jump.

Thus we assume that e is composed almost exclusively of turbulent energy. Furthermore, we will assume that the turbulence is roughly isotropic. Notice that it is not *a priori* clear whether the number of dimensions over which the turbulent energy is partitioned should be taken to be two or three. It is conceivable that there is a two-dimensional component to the turbulence, associated with the main vortex of the jump, and a comparable three-dimensional component, resulting from energy equipartition at smaller scales. Thus we will leave the dimensionality of the turbulence open; it is likely that an intermediate number between two and three best represents real flows. As it turns out, the sensitivity of our predictions to dimensionality is fairly small.

In short, we assume that the internal energy density (per unit mass and unit height) of the fluid has the form

$$e = d\left(\frac{1}{h}\int_0^h \tfrac{1}{2}\langle w^2\rangle \,\mathrm{d}z\right), \tag{4.1}$$

where d is the number of dimensions (somewhere between two and three), w is the vertical component of the velocity, z is the vertical coordinate, and $\langle w^2\rangle$ indicates the average of w^2 over the turbulence space and time scales.

We shall now show that simple physical arguments allow us to bound this turbulent energy e by an expression of the form

$$0 \leqslant e \leqslant \frac{d}{4}bh. \tag{4.2}$$

This constitutes our turbulent partial closure: a bound on the amount of turbulent energy that a layer of fluid may contain without immediately mixing further. A way to see this that we find particularly insightful is through a thought experiment (as an aside, such an experiment can be realized in the laboratory with present fluid measurement technology, something that we plan to do in the near future).

Consider a bucket containing a homogeneous fluid, and assume that we have devices to excite turbulent motion in the fluid interior. Our contention is that the amount of turbulence at a given depth z cannot be made arbitrarily large without the fluid spilling out. This is because the pressure within the turbulent region of the fluid may exceed the weight of the fluid above it. To see this, consider a horizontal plane cutting through the fluid at depth z. The pressure on this plane needs to balance exactly the weight of fluid above, else the fluid will accelerate. The physical origin of this pressure lies in three distinct sources: intermolecular repulsive forces, which oppose compression; momentum transfer by thermal molecular motion; and macroscopic momentum transfer by parcels of fluid in turbulent motion. This last turbulent component to the pressure, P_T, is given by

$$P_T = \rho\langle w^2\rangle. \tag{4.3}$$

This is entirely analogous to the conceptualization of pressure in kinetic theory of gases, as the momentum transfer by molecular thermal motion. In our case, the momentum flux through the plane will be given by the momentum itself ρw times its transfer rate w, giving rise to (4.3). If P_T should exceed the weight $g\rho z$ of fluid above the plane, this fluid will accelerate up and detach from the fluid below, since the other two components of the pressure cannot be negative. Hence we need to have

$$\langle w^2 \rangle \leqslant gz. \tag{4.4}$$

When the turbulent fluid layer in the bucket underlies another layer with lighter fluid, the argument leading to (4.4) remains valid if one replaces the gravity constant g by the reduced gravity $b = g\Delta\rho/\rho$. Hence (4.4) becomes

$$\langle w^2 \rangle \leqslant bz. \tag{4.5}$$

Placing (4.5) into (4.1), we obtain the inequality (4.2).

The bound in (4.2) seems to agree rather well with the available experimental evidence. Figure 16 of Pawlak & Armi's paper on entrainment in stratified currents (Pawlak & Armi 2000) depicts a 'RMS Froude number' which, in our notation, is defined by

$$F_{rms} = \sqrt{\frac{\overline{(u-\bar{u})^2}}{bh}}, \tag{4.6}$$

where the bars indicate averages in the same sense used in equation (4.1); i.e. both over the thickness of the layer and over the turbulence scales. Using our assumption of turbulence isotropy, the definition in (4.1), and the bound in (4.2), we obtain

$$F_{rms} = \sqrt{\frac{\overline{w^2}}{bh}} = \sqrt{\frac{2e}{dbh}} \leqslant \sqrt{\frac{1}{2}} \approx 0.71. \tag{4.7}$$

This upper bound agrees quite well with the measured values for F_{rms} in the region of high entrainment rate, just downstream of the experimental sill. Further downstream, in a region of highly reduced entrainment, this value is decreased roughly by a factor of two, to $F_{rms} \approx 0.35$.

It seems likely that, in the neighbourhood of a very strong hydraulic jump, the right-hand inequality in (4.2) should become an equality. The reasons are different for the flows immediately upstream and immediately downstream of the jump. The former, being supercritical, typically downslope, and highly turbulent, would have been entraining ambient fluid before encountering the jump. Hence it would be in a state right at the critical value for the turbulence discussed earlier and leading to (4.2). On the other hand, the level of turbulence in the flow immediately downstream is determined by the jump, which presumably would tend to maximize the conversion of the kinetic energy of the upstream flow into turbulence and entrainment. We will postpone any consideration of this scenario of maximally turbulent flow to § 6, after fully exploring, in § 5, the consequences of the inequality in (4.2).

5. Bounds following from the partial closure

For the reader more interested in the physics of internal hydraulic jumps than in their mathematical analysis, we emphasize that all of our modelling assumptions have been made at this point. The manipulations in this section, leading from these assumptions to the bounds summarized in table 1 from § 1, are purely algebraic, and do not contain any hidden extra physical hypotheses.

With the partial closure (4.2) in hand, we can revisit the jump conditions. We recall that b, u and h are non-dimensional ratios; see (3.5). Equation (3.4) becomes

$$\tfrac{1}{2}h_2u_2^3 + b_2h_2^2u_2 + h_2u_2e_2 = \tfrac{1}{2}h_1u_1^3 + b_1h_1^2u_1 + h_1u_1e_1.$$

Dividing by $\frac{1}{2}h_1u_1^3$, and using the fact that $b_2h_2u_2 = b_1h_1u_1$, we obtain

$$\frac{2}{F_1^2}(h-1) + 2hu\frac{e_2}{u_1^2} = 1 - hu^3 + 2\frac{e_1}{u_1^2};$$

here, F_1 is the Froude number upstream. Similarly, we divide both sides of the jump condition (3.3) by $h_1u_1^2$. The full set of jump conditions becomes

$$bhu = 1,$$

$$\frac{1}{2}\left(\frac{h}{u} - 1\right)\frac{1}{F_1^2} + hu^2 - 1 = 0,$$

$$\frac{2}{F_1^2}(h-1) + \frac{2}{u_1^2}(hue_2 - e_1) = 1 - hu^3.$$

Solving for F_1^2 in the second equation and for the energy terms in the third equation, we obtain

$$bhu = 1, \tag{5.1}$$

$$F_1^2 = \frac{h-u}{2u(1-hu^2)} = \frac{1-bu^2}{2u^2(b-u)}, \tag{5.2}$$

$$\frac{4F_1^2}{u_1^2}(b-u)\left(\frac{e_2}{b} - e_1\right) = \left(u + \frac{1}{u} - 2\right)^2 + 3\left(b + \frac{1}{b} - 2\right). \tag{5.3}$$

The partial closure inequality (4.2) applied to the flow downstream of the jump can be written as

$$0 \leqslant \frac{4F_1^2}{u_1^2}ue_2 \leqslant d. \tag{5.4}$$

Since $e_1 \geqslant 0$, the left-hand side of (5.3) can be bounded, using (5.4), as follows:

$$\frac{4F_1^2}{u_1^2}(b-u)\left(\frac{e_2}{b} - e_1\right) \leqslant \frac{4F_1^2}{u_1^2}\frac{b-u}{b}e_2 \leqslant d\frac{b-u}{bu} = d\left(\frac{1}{u} - \frac{1}{b}\right). \tag{5.5}$$

From (5.3) and (5.5), we obtain

$$\left(u + \frac{1}{u} - 2\right)^2 + 3\left(b + \frac{1}{b} - 2\right) \leqslant d\left(\frac{1}{u} - \frac{1}{b}\right),$$

which can also be written as

$$3b + \frac{d+3}{b} \leqslant 4u + \frac{d+4}{u} - u^2 - \frac{1}{u^2}. \tag{5.6}$$

Using the fact that $u, b \leqslant 1$ and the inequality (5.6), it follows that $u \leqslant b$, since the right-hand side of (5.6) can be bounded from above by $3u + (d+3)/u$. Thus, the implicit constraint in (5.2), i.e. $b \geqslant u$, is satisfied so long as inequality (5.6) holds.

We also have to impose the entropy conditions (3.7), i.e.

$$u \leqslant 1, \qquad h \geqslant 1, \qquad b \leqslant 1. \tag{5.7}$$

We conclude that all the conditions deriving from the conservation equations and our partial closure that b, u and h have to satisfy are (5.1), (5.2), (5.6) and (5.7).

One final remark: it is easy to see, from the jump conditions, that the internal energy flow across a hydraulic jump can only increase, i.e. that

$$[hue] > 0.$$

This follows by noticing that the right-hand side of (5.3) is positive if a jump exists, i.e. if $u < 1$, $b < 1$. Then,

$$\frac{e_2}{b} - e_1 > 0.$$

From (5.1), it results that

$$\frac{e_2}{b} - e_1 = uhe_2 - e_1 = \frac{[hue]}{u_1 h_1},$$

and therefore $[hue] > 0$.

5.1. *Bounds on b, u and h*

The inequality (5.6) turns out to be very important in establishing relevant bounds for b, u and h, i.e. lower bounds for b and u and an upper bound for h. To further analyse the right-hand side of (5.6), we introduce the function

$$g : (0, 1] \to R, \quad g(u) = 4u + \frac{d+4}{u} - u^2 - \frac{1}{u^2}.$$

On the interval $(0, 1]$, g is concave and has one global maximum, denoted by M. Then, from (5.6), it follows that

$$3b + \frac{d+3}{b} \leqslant M. \tag{5.8}$$

Since the left-hand side of (5.8) is a decreasing function of b on the interval $(0, 1]$, the minimum possible value of b is achieved when equality is realized in (5.8). Using Newton's method to compute M, and then solving the quadratic equation associated with (5.8), we find the minimum value of b. For $d = 2$, we obtain

$$b \geqslant 0.588.$$

Equality is realized for $u = 0.358$ and $h = 4.738$. Here, and throughout the rest of the paper, the numerical results are truncated after the third decimal digit. For $d = 3$, we obtain

$$b \geqslant 0.508,$$

with equality realized for $u = 0.298$ and $h = 6.589$.

We now turn our attention to finding a lower bound on u. The left-hand side of (5.6) is a decreasing function of b on $(0, 1]$. Therefore,

$$d + 6 \leqslant g(u).$$

On the interval $(0, 1]$, the function $g(u)$ increases from $-\infty$ to M and then decreases to $d + 6$. The minimum value of u can be obtained by solving $g(u) = d + 6$ and eliminating the solution $u = 1$. Using Newton's method, we obtain, for $d = 2$, that

$$u \geqslant 0.229.$$

Equality is realized for $b = 1$ and $h = 4.365$. For $d = 3$, we obtain

$$u \geqslant 0.182,$$

with equality realized for $b = 1$ and $h = 5.486$.

To establish an upper bound on h, we use the fact that $b = 1/uh$ and rewrite (5.6) as

$$h^2u(d+3) - h\left(4u + \frac{d+4}{u} - u^2 - \frac{1}{u^2}\right) + \frac{3}{u} \leqslant 0. \tag{5.9}$$

In other words

$$h \leqslant \max_{u \in (0,1]} h_2,$$

where h_2 is the largest of the two solutions of the quadratic equation associated with (5.9). We use once again Newton's method and obtain, for $d = 2$, that

$$h \leqslant 5.572.$$

Equality is realized for $b = 0.654$ and $u = 0.274$. For $d = 3$, we obtain

$$h \leqslant 7.772,$$

with equality realized for $b = 0.567$ and $u = 0.226$.

6. A full closure for the energy

In this section, we develop a closure for strong hydraulic jumps based on the assumption that the upper bound in (4.2) is in fact an equality. As mentioned in § 4, for strong hydraulic jumps, the flows both upstream and downstream of the jump should be maximally turbulent: the former due to its highly supercritical nature; the latter so as to maximize the irreversibility of the jump. This hypothesis seems to agree rather well with the observations reported in Pawlak & Armi (2000) for the initial highly entraining region of a dense overflow; see the discussion at the end of § 4. We now proceed to explore the consequences of this closure hypothesis.† Its full validation, of course, should come from detailed observations of real overflows, of the kind that state of the art oceanography appears to be ready to obtain. In fact, a motivation for pursuing a full closure is that it allows us to make sharp quantitative predictions of the relation between the various variables involved in an internal hydraulic jump. These predictions, developed in § 6.3, are ideally suited to test our theory both in the real ocean and in the laboratory.

Throughout this section, we assume the following form for the internal energy immediately upstream and downstream of the jump:

$$e_1 = \frac{d}{4}b_1h_1 \quad \text{and} \quad e_2 = \frac{d}{4}b_2h_2. \tag{6.1}$$

Equation (6.1) can be recast as

$$\frac{4F_1^2}{u_1^2}e_1 = d \quad \text{and} \quad \frac{4F_1^2}{u_1^2}e_2 = \frac{d}{u}.$$

Moreover,

$$\frac{4F_1^2}{u_1^2}(b-u)\left(\frac{e_2}{b} - e_1\right) = (b-u)\left(\frac{d}{bu} - d\right) = d\left(u + \frac{1}{u} - b - \frac{1}{b}\right).$$

† If mathematical beauty and compactness is to be taken as a sign that the underlying physical theory holds some degree of truth (as it has so often been the case through the history of science), then we hope that the reader will agree with us, after reading this section, that our hypothesis may not be completely at odds with reality.

Then, the energy equation (5.3) becomes

$$d\left(u+\frac{1}{u}-b-\frac{1}{b}\right)=\left(u+\frac{1}{u}-2\right)^2+3\left(b+\frac{1}{b}-2\right). \quad (6.2)$$

Employing once again the notation $x = b + 1/b$ and $y = u + 1/u$, we can recast (6.2) as

$$d(y-x)=(y-2)^2+3(x-2). \quad (6.3)$$

Since $u, b \leqslant 1$ and $x \geqslant 2$, it follows from (6.3) that $y \geqslant x$ and therefore $b \geqslant u$. In other words, the implicit constraint that $b - u \geqslant 0$ from (5.2) is satisfied. Thus b, u and h only have to satisfy (5.1), (5.2) and (5.7), in addition to (6.2).

6.1. *Bounds on b, u and h*

We now derive relevant bounds for b, u and h. Due to the simplified form (6.3) of the energy equation, it is possible to compute the lower bounds for b and u without making use of Newton's method. Solving for x in (6.3), we obtain

$$x=\frac{2+(d+4)y-y^2}{d+3}. \quad (6.4)$$

The denominator of the right-hand side of (6.4) reaches its maximum at $y = (d+4)/2$, so

$$x \leqslant 2+\frac{d^2}{4(d+3)}.$$

We recall that $x = b + 1/b$. Then, for $d = 2$, we obtain

$$b \geqslant 0.641,$$

with equality realized for $u = 0.382$ and $h = 4.079$. For $d = 3$, we obtain

$$b \geqslant 0.547,$$

with equality realized for $u = 0.313$ and $h = 5.824$.

Since $x \geqslant 2$, it follows from (6.4) that

$$y^2-(d+4)y+2d+4=(y-2)(y-2-d)\leqslant 0,$$

and therefore that

$$y=u+\frac{1}{u}\leqslant d+2.$$

Let u_{min} be the only solution of $u+1/u = d+2$, so that $u_{min} \leqslant 1$. For $d = 2$, we obtain

$$u \geqslant u_{min} = 0.267.$$

Equality is realized for $b = 1$ and $h = 3.732$. For $d = 3$, we obtain

$$u \geqslant u_{min} = 0.208.$$

Equality is realized for $b = 1$ and $h = 4.791$.

The derivation of the upper bound for h is more complicated. Replacing b by $1/hu$ in (6.2), we obtain a quadratic equation in h:

$$h^2u-\frac{h}{d+3}\left((d+4)\left(u+\frac{1}{u}\right)-u^2-\frac{1}{u^2}\right)+\frac{1}{u}=0. \quad (6.5)$$

In other words,

$$h \leqslant \max_{u \in (0,1]} h_2,$$

where h_2 is the largest of the two solutions of the quadratic equation associated with (6.5). Using Newton's method, we obtain, for $d = 2$, that

$$h \leqslant 4.705.$$

Equality is realized for $u = 0.303$ and $b = 0.701$. For $d = 3$, we obtain

$$h \leqslant 6.767,$$

with equality realized for $u = 0.245$ and $b = 0.603$.

6.2. *Upper and lower bounds on the Froude numbers*

In this section, we derive upper and lower bounds for the Froude numbers upstream and downstream of the jump. A note here is appropriate on the meaning of these numbers. In regular hydraulics, the Froude number is the ratio of the mean speed of the flow to the characteristic velocity at which information propagates. The flow upstream of a jump should be supercritical, i.e. it must have $F_1 > 1$, and the flow downstream should be subcritical, with $F_2 < 1$, so that the right amount of information reaches the jump through characteristics from both sides. This need not be the case, however, for two-layer miscible flows. The reason is that the characteristic speed for flows involving mixing is not necessarily given by $\sqrt{bh}$. In order to compute characteristic speeds, we need to know the equations describing the physics away from the jump, and these depend on how the mixing process is described. Only under the assumption that no mixing takes place away from jumps will the Froude numbers defined in (3.6) have the same meaning as in regular hydraulics. In this paper, we are only concerned with the jump conditions at internal hydraulic jumps, not with the partial differential equations away from them, and so we have no control over the characteristic speeds on either side of the jump. However, the Froude numbers are still important reference values in internal hydraulics, and so we proceed to determine what our closure implies for them. Surprisingly, one of these implications is that indeed $F_1 > 1$, just as in regular hydraulics, even though the equations describing the flow away from the jump are still to be determined. On the other hand, F_2 is not necessarily bounded from above by 1. We will derive achievable upper and lower bounds for both F_1 and F_2.

At first, it may appear puzzling that bounds on the Froude numbers exist: to our knowledge, they have no analogues in either open channel hydraulics or gas dynamics. In principle, it can be argued that an experimentalist has absolute freedom to select the Froude number upstream, which therefore cannot possibly have an upper bound. However, for an arbitrarily set upstream Froude number, a standing hydraulic jump need not exist. The bound we derive below suggests that, if an upstream Froude number is specified above the bound, any jump that develops will be swept downstream by the flow, i.e. it will not be steady.

In order to derive the aforementioned bounds, we recall the expression (5.2) for F_1, obtained from the reduced form of the momentum equation:

$$F_1^2 = \frac{1 - bu^2}{2u^2(b-u)}.$$

Equation (6.2) is quadratic in b. Let $\beta(u)$ be the smaller of its two solutions, i.e.

$b = \beta(u) \leqslant 1$. Then F_1 can be regarded as a function of u, given by

$$F_1(u) = \sqrt{\frac{1-\beta(u)u^2}{2u^2(\beta(u)-u)}}. \tag{6.6}$$

On the interval $[u_{min}, 1]$, the function $F_1(u)$ has exactly one maximum point, which can be found using Newton's method. The following upper bounds for F_1 are thus obtained: for $d = 2$,

$$F_1 \leqslant 3.595,$$

with equality achieved for $b = 0.673$, $u = 0.318$, and $h = 4.658$; and, for $d = 3$,

$$F_1 \leqslant 4.751,$$

with equality achieved for $b = 0.581$, $u = 0.255$, and $h = 6.721$.

Moreover, the minimal value of $F_1(u)$ is achieved in the limit as $u \rightarrow 1$, and, therefore, as $b, h \rightarrow 1$. Then,

$$F_1 \geqslant \lim_{u\rightarrow 1} F_1(u) = \left(\frac{d+2+\sqrt{d(d+3)}}{2}\right)^{1/2}.$$

For $d = 2$,

$$F_1 \geqslant 1.892,$$

with equality achieved in the limit as u, b, $h \rightarrow 1$. For $d = 3$,

$$F_1 \geqslant 2.149,$$

with equality once again achieved in the limit as u, b, $h \rightarrow 1$.

Establishing bounds for F_2 can be done similarly. From (3.6) and (5.2), we obtain that $F_2^2 = u^3 F_1^2$, and therefore, from (6.6), it follows that

$$F_2(u) = \sqrt{\frac{u(1-\beta(u)u^2)}{2(\beta(u)-u)}}. \tag{6.7}$$

The function $F_2(u)$ is increasing on the interval $[u_{min}, 1]$. Thus, the smallest possible value for F_2 corresponds to $u = u_{min}$. For $d = 2$,

$$F_2 \geqslant 0.412,$$

with equality realized for $b = 1$, $u = 0.267$, and $h = 3.732$. For $d = 3$,

$$F_2 \geqslant 0.355,$$

with equality realized for $b = 1$, $u = 0.208$, and $h = 4.791$.

The largest value for F_2 corresponds once again to the limit case when $u \rightarrow 1$, and is therefore equal to the smallest values for F_1, i.e.

$$F_2 \leqslant \lim_{u\rightarrow 1} F_1(u) = \left(\frac{d+2+\sqrt{d(d+3)}}{2}\right)^{1/2}.$$

For $d = 2$, $F_2 \leqslant 1.892$, and, for $d = 3$, $F_2 \leqslant 2.149$, with equality in both cases achieved in the limit as $u, b, h \rightarrow 1$.

6.3. *Sharp predictions*

Within our full closure for internal hydraulic jumps, it is possible to predict sharp values for all the flow ratios u, b and h across the jump, as well as the Froude

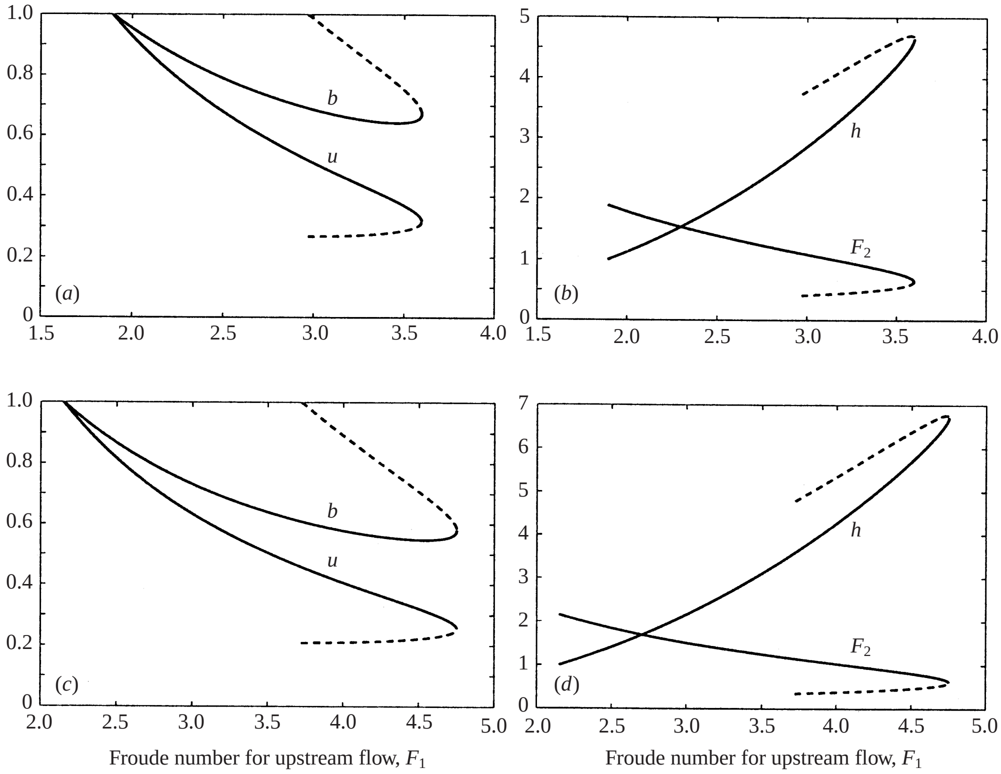

FIGURE 3. Dependence of u, b, h and F_2 on F_1 for: (a,b) $d = 2$ and (c,d) $d = 3$.

number F_2 downstream of the jump, as functions of the upstream Froude number F_1. Such predictions, described in this subsection, provide an ideal testing ground for experimental and observational checks (and refinements) of our closure assumptions.

The derivation of these values follows the following path. First, we use (6.6) to compute the velocity ratio u as a function of the upstream Froude number F_1. Next we compute the buoyancy ratio b from $b = \beta(u)$, where $\beta(u)$, introduced above, is the smallest of the two solutions to (6.2). The height ratio h follows from $h = 1/bu$. Finally, from (6.7), we obtain the value of F_2 as a function of F_1. These results are graphically represented in figure 3, for both $d = 2$ and $d = 3$.

We have dotted the branch of these predictions starting at the maximum value of F_1 and ending at a jump with finite strength but no mixing; i.e. $b = 1$. Even though this branch of results follows formally from our full closure, it is probably not entirely consistent with its strong hypotheses. In particular, our assumption of maximal turbulence corresponds to a situation where all excess turbulent energy has been used for entraining upper fluid. It is clearly hard to reconcile this picture with the non-entraining, yet strong jump lying at the end of the dotted line in figure 3.

We are not equally concerned about the other end of the predictions, where $b = h = u = 1$, corresponding to no jump. Even though the jumps in the neighbourhood of this point are necessarily weak, they still correspond to a highly turbulent situation, even though most of this turbulence is not generated by the jump, but already present in the conditions upstream. In fact, this highly turbulent and entraining character of the flow is responsible for the curve of $F_2(F_1)$ not starting at $F_1 = F_2 = 1$, but at a much higher value.

7. Discussion

In this paper, we propose two closures for strong, highly entraining, internal hydraulic jumps in two-layer flows. The first is only a partial closure, resulting from an upper bound on the amount of turbulent energy that the flow can admit. The second, a full closure, applies to highly turbulent flows, where the upper bound on the energy can be made into an equality. In this work we make the following assumptions:

(*a*) The flow away from the jump is well-described by a two-layer shallow-water theory. This entails two main assumptions. The first is that the flow away from the jump is in hydrostatic balance. There is little question that, in most geophysical applications, this is a good approximation. The second assumption is that both the velocity and the density remain nearly uniform in both layers downstream of the jump. The physical intuition behind this is that the upper layer is not greatly affected by the jump, while the lower one is rapidly homogenized by turbulent mixing.

(*b*) In the neighbourhood of the jump, most of the energy dissipated by the mean flow goes into isotropic turbulence, since: (i) for miscible fluids, there is no surface tension energy; (ii) for strong, highly turbulent hydraulic jumps, turbulence tends to average away both radiating waves and organized shear; and (iii) the time scale for the conversion of turbulent energy into heat is much longer than the time spent by most fluid particles in a neighbourhood of the jump. Note that by neighbourhood here we mean distances of the order of tens of jump widths.

(*c*) The amount of turbulence in hydrostatic stratified flows is limited by the buoyancy. If the turbulence becomes too high, the various fluid layers will completely mix. Thus, the very existence of distinct layers ensures that a critical value is not surpassed.

(*d*) For the full version of the closure we further assume that the bound on the turbulence is, in fact, achieved both upstream and downstream of the jump. In other words, we assume that the flow is maximally turbulent near the jump.

Out of these closure hypotheses, surprising bounds emerge for the velocity, the height, and (more significantly) the buoyancy ratios across the jump. Depending on the details of the closure, the last bound ranges from $0.5 < b \leqslant 1$ to $0.64 < b \leqslant 1$. Strong hydraulic jumps should yield values of b close to the lower bounds, so we predict the volume flow in the lower layer to increase by 50% to a 100% across them. This is consistent with flow measurements for both the Mediterranean outflow and the Denmark Strait overflow, which roughly double within a few hundred kilometres of the straits – with some evidence that these volume increases occur across very localized areas of strong mixing (O'Neil Baringer & Price 1997*a*; Saunders 2001).

Another consequence of the full closure, at least from a theoretical point of view, is that it implies the existence of upper and lower bounds for the Froude numbers upstream and downstream of the jump, respectively. This is interesting because no such bounds exist for 'regular' hydraulic jumps in a single fluid layer.

We eagerly await the validation of our theory both by laboratory experiments and by oceanographic and atmospheric observations:

> "Grau, teurer Freund, ist alle Theorie, und grün des Lebens goldner Baum." (Goethe).

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